Design of Length-Compatible Low-Density Parity-Check Codes

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Abstract

Length-compatible low-density parity-check (LC-LDPC) codes are a class of low-density parity-check (LDPC) codes which can support a wide range of lengths for a given rate. They may be obtained by applying shortening and puncturing schemes to good LDPC codes. However, this conventional approach does not always guarantee them to have a good performance because their degree distributions may be wildly varied. In this paper we propose a novel algorithm to generate a mother code for LC-LDPC codes of a given rate such that their degree distributions are almost the same as the degree distribution of the mother code. Numerical results show that LC-LDPC codes constructed by our approach perform much better than those by the conventional approach.

Index Terms

Low-density parity-check (LDPC) codes, progressive edge-growth (PEG) algorithm, length compatibility, degree distributions.

I. INTRODUCTION

Low-density parity-check (LDPC) codes, first introduced by Gallager [1], achieve the performance near to the Shannon limit by iterative decoding [2]-[4]. They are extensively employed in communication system standards such as DVB-S2 [5], IEEE 802.16e [6], and IEEE 802.11n [7] because of their capacity-approaching performance and low decoding complexity. In these systems, a number of LDPC codes of various lengths are required even for a fixed rate in

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communication systems. Furthermore, for simplicity of implementation, it is more desirable to support such lengths using a single encoder and decoder. For these reasons, it is an interesting and important problem to construct a parity-check matrix which can support LDPC codes with a wide range of lengths for a given rate. These codes will be referred to as length-compatible LDPC (LC-LDPC) codes. They are distinct from rate-compatible LDPC codes [8]-[12] in the sense that the latter support various rates for a fixed information length.

Information-shortening and parity-puncturing schemes have been extensively studied in order to get good LDPC codes from a given LDPC code [8]-[13]. In general, shortening schemes reduce the rate, while puncturing schemes increase it. Therefore, a common way to obtain LC-LDPC codes from a good mother code is to combine these two schemes in a proper way. However, this conventional approach does not always guarantee them to have a good performance because their degree distributions may be wildly changed.

In this paper we propose a novel algorithm to construct LDPC codes with dual-diagonal parity structure, which may be used as a mother code for LC-LDPC codes. The dual-diagonal parity structure, i.e., the accumulator, is one of the most commonly used structures for linearly encodable LDPC codes which perform as well as random LDPC codes [12]-[17]. Furthermore, it is preserved even when parity-puncturing schemes are applied, because any two check nodes connected to a punctured parity node of degree two can be replaced by a single merged super-check node [13]. For a given rate, the proposed algorithm generates a mother code for LC-LDPC codes with a wide range of lengths such that their degree distributions are almost the same as the degree distribution of the mother code. Numerical results show that LC-LDPC codes obtained from the mother code constructed by the proposed algorithm perform much better than those derived from the mother code constructed by the progressive-edge-growth (PEG) algorithm.

The outline of the paper is as follows. In Section II, we review shortening/puncturing schemes and present a problem formulation. Section III presents an algorithm for constructing a mother code for LC-LDPC codes. Numerical results are given in Section IV. Finally, we give concluding remarks in Section V.
II. Preliminaries and Problem Formulation

A. Shortening and Puncturing Schemes

Given an LDPC code of length $n$ and dimension $k$, consider the set of codewords whose $s$ information bits are equal to zero. It is a $(k-s)$-dimensional subcode. If the $s$ information bits are deleted from each of these codewords, we obtain an LDPC code of length $n-s$ and dimension $k-s$. This procedure is called information-shortening (shortening for short). On the other hand, some particular bits of every transmitted codeword can be omitted before transmission in order to increase the effective rate of the code. Usually, this procedure is applied to parity bits and is called parity-puncturing (puncturing for short).

In general, the degree distribution of an LDPC code may be varied by information-shortening since the columns corresponding to the shortened part are removed from the parity-check matrix. It may also be changed by parity-puncturing since any two check nodes connected to a punctured variable node of degree two can be replaced by a single merged super-check node [13]. In particular, the Tanner graph with punctured parity bits can be simplified to an equivalent Tanner graph without punctured parity bits. Throughout the paper, we are restricted only to LDPC codes with dual-diagonal structure [12]-[17].

B. Problem Formulation

As mentioned in Introduction, the length compatibility of LDPC codes can be achieved by employing shortening and puncturing schemes appropriately. Given an LDPC code of length $n$ and dimension $k$ as a mother code, the lengths to be shortened and punctured, denoted by $s$ and $p$, respectively, are determined by the target rate $R$ and the target information length $N_I$. Since the rate is assumed to be fixed for any target information length $N_I$, we have

$$R = k/n, \quad N_I = k - s,$$

$$N_I/R = n - (s + p).$$

(1)

The left-hand side of (1) means the length of a derived LDPC code of information length $N_I$ and rate $R$, while the right-hand side is the number of bits to be transmitted after shortening and puncturing. Formally, the resulting LDPC code will be referred to as an LC-LDPC code throughout the paper.
The degree distributions obtained by density evolution [3] perform well even for moderate-length LDPC codes, although they are designed for infinite-length LDPC codes. This implies that LC-LDPC codes may perform well when the degree distribution of a mother code found by density evolution is still preserved after shortening and puncturing schemes are applied. For this reason, our goal is to design a mother code such that its degree distribution is preserved in the process of generating LC-LDPC codes with a wide range of lengths. Consider an ensemble of LDPC codes with degree distribution \( (\lambda(x), \rho(x)) \), where

\[
\lambda(x) = \sum_{i=2}^{d_v} \lambda_i x^{i-1}, \quad \rho(x) = \sum_{i=2}^{d_c} \rho_i x^{i-1},
\]

where \( \lambda_i (\rho_i) \) is the fraction of edges connected to variable (check, respectively) nodes of degree \( i \) and \( d_v (d_c) \) is the maximum variable (check, respectively) node degree. Given the degree distribution \( (\lambda(x), \rho(x)) \) of the mother code, let \( \hat{\lambda}(S, P, x) \) (\( \hat{\rho}(S, P, x) \)) be the variable (check, respectively) degree distribution of an LC-LDPC code obtained from information-shortening by length \( s = Sk \) and parity-puncturing by length \( p = Pm \), where \( 0 \leq S, P < 1 \) and \( m \) is the parity length of the mother code given by \( m = n - k \). Since the rate of any LC-LDPC code is fixed as that of the mother code, we get \( S = P \). Therefore, the problem to design LC-LDPC codes of a fixed rate \( R \) is to construct a mother code such that

\[
\hat{\lambda}(S, S, x) = \lambda(x) \quad \text{and} \quad \hat{\rho}(S, S, x) = \rho(x)
\]

over a wide range of \( S \).

III. CONSTRUCTION OF A MOTHER CODE FOR LC-LDPC CODES

The proposed method to construct a mother code for LC-LDPC codes is based on splitting and appending. In the Tanner graph of an LDPC code with dual-diagonal parity structure, the \( i \)th parity node \( p_i \) is connected to the \( i \)th and \((i+1)\)st check nodes denoted by \( x_i \) and \( x_{i+1} \), respectively. These two check nodes can be merged into a super-check node \( X_j \) when \( p_i \) is punctured. This merging procedure is denoted by \( (x_i, x_{i+1}) \rightarrow X_j \). Splitting is the inverse operation of merging due to puncturing and the order of the check nodes to be split is determined by a puncturing order. For example, assume that puncturing is performed in the order \((p_3; p_5; p_1; p_7; p_3)\). Then
the merging order is
\[(x_9, x_{10}) \rightarrow X_5 ; (x_5, x_6) \rightarrow X_4 ; (x_1, x_2) \rightarrow X_3 ; \]
\[(x_7, x_8) \rightarrow X_2 ; (x_3, x_4) \rightarrow X_1), \]
and the splitting order is
\[(X_1 \rightarrow (x_3, x_4) ; X_2 \rightarrow (x_7, x_8) ; X_3 \rightarrow (x_1, x_2) ; \]
\[X_4 \rightarrow (x_5, x_6) ; X_5 \rightarrow (x_9, x_{10})].\]

On the other hand, appending is the inverse operation of shortening. A submatrix corresponding to additional information bits is appended to the given parity-check matrix. This submatrix may be generated by the PEG algorithm [18].

The proposed construction methodology focuses on how to preserve the degree distribution of LC-LDPC codes, when the shortening and puncturing rules are applied to a mother code. In this paper, the following shortening and puncturing rules are assumed to be applied.

- Shortening begins with the last part of the information bits.
- Punctured parity bits are selected so that the maximum number of the steps required to recover the punctured bits is minimized [8], [12], [11].

Based on these rules, the shortening order and the puncturing order can be determined. Furthermore, the variable degree distribution of the shortened block should be the same as that of the information part of the mother code in order to preserve the variable degree distribution of LC-LDPC codes. Consequently, the variable degree distributions for appended submatrices as well as the splitting order are predetermined for the proposed method. Let \( \{N_I(1), \ldots, N_I(L)\} \) be the set of target information lengths where \( N_I(i) < N_I(j) \) for any \( i < j \) and \( L \) is the number of distinct target information lengths. The parity length \( N_P(i) \) corresponding to the information length \( N_I(i) \) is given by \( N_P(i) = \lceil N_I(i)/R \rceil - N_I(i) \) for a given rate \( R \).

The proposed method to construct a mother code for LC-LDPC codes is summarized as follows:

- **Step 1**: Construct a Tanner graph for an LDPC code of information length \( N_I(1) \) and parity length \( N_P(1) \) by using the PEG algorithm (Fig. 1(a)). Set \( j = 1 \).
- **Step 2**: Split \( N_P(j + 1) - N_P(j) \) check nodes so that a Tanner graph with information length \( N_I(j) \) and parity length \( N_P(j + 1) \) is constructed (Fig. 1(b)).
Step 3: Append $N_I(j+1) - N_I(j)$ information nodes to the last part of the information nodes so that a Tanner graph with information length $N_I(j+1)$ and parity length $N_P(j+1)$ is constructed (Fig. 1(e)).

Step 4: Establish the edges connected to the appended information nodes by the PEG algorithm (Fig. 1(d)). Increase $j$ by 1.

Step 5: Go to Step 2 until $j$ reaches $L$.

Step 6: Output a Tanner graph with information length $N_I(L)$ and parity length $N_P(L)$ as a mother code for LC-LDPC codes.

IV. Numerical Results

The bit-error rate (BER) performances of LC-LDPC codes obtained from the mother codes constructed by the proposed algorithm and the PEG algorithm, respectively, are compared over an additive white Gaussian noise (AWGN) channel, where the sum-product algorithm with at most 100 iterations is employed. The degree distribution for the mother codes of rate 0.5 is chosen to be

$$\lambda(x) = 0.30780x + 0.27287x^2 + 0.41933x^6$$

and

$$\rho(x) = 0.4x^5 + 0.6x^6$$

in [16]. For the given information length $k$ of a mother code, the set of target information lengths of LC-LDPC codes is set to \{0.2$k$, 0.4$k$, 0.6$k$, 0.8$k$, $k$\}. For simple notation, ‘LC-LDPC-I’ and ‘LC-LDPC-II’ codes denote LC-LDPC codes obtained from the mother codes constructed by the proposed algorithm and the PEG algorithm, respectively.

The degree distributions and noise thresholds of LC-LDPC-I and LC-LDPC-II codes of rate 0.5 with $k = 10000$ are compared in Table I. Note that the degree distributions of LC-LDPC-II codes are wildly varied, while LC-LDPC-I codes have almost the same degree distributions as their mother code. As a result, LC-LDPC-I codes have almost the same noise thresholds as their mother code, which are much better than those of LC-LDPC-II codes for the considered target information lengths 8000, 6000, 4000, and 2000.

Fig. 2 shows the BER performances of LC-LDPC-I and LC-LDPC-II codes listed in Table I. For the considered target information lengths, LC-LDPC-I codes have much better performance than LC-LDPC-II codes. In particular, the LC-LDPC-I code of $N_I = 6000$ performs about 0.54 dB better than the LC-LDPC-II code of the same length at BER = $10^{-4}$, as can be expected from the difference between their noise thresholds. As another example, the BER performances of LC-LDPC-I and LC-LDPC-II codes with $k = 2000$ are given in Fig. 3. Their BER performances
have the same tendency as in the case of \( k = 10000 \).

More extensive design examples of LC-LDPC codes have been tested over a variety of rates and lengths. For example, Fig. 4 shows the performance of LC-LDPC codes of rate 0.82 for target information lengths \( \{2000, 4000, 6000, 8000, 10000\} \) and Fig. 5 shows the performance of LC-LDPC codes of rate \( 1/3 \) for target information lengths \( \{1080, 2160, 3240, 4320, 5400\} \). Their degree distributions are based on [16] and [5], respectively. These numerical results also show that LC-LDPC codes constructed by our approach perform much better than those by the conventional approach even for high and low rates.

V. CONCLUDING REMARKS

Based on splitting and appending, we proposed a new algorithm to construct a mother code for LC-LDPC codes such that their degree distributions are almost the same as the degree distribution of the mother code over a wide range of lengths. These codes may be applicable to next-generation communication systems which requires to support various lengths for a given rate using a single encoder and decoder.

REFERENCES


TABLE I  
THE DEGREE DISTRIBUTIONS AND NOISE THRESHOLDS OF LC-LDPC CODES DERIVED FROM MOTHER CODES OF LENGTH $k = 10000$ AND RATE 0.5

<table>
<thead>
<tr>
<th>Code</th>
<th>$S$</th>
<th>$N_I$</th>
<th>$\hat{\lambda}(S, x)$</th>
<th>$\hat{\rho}(S, x)$</th>
<th>Noise threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>deg. 2     deg. 3     deg. 7</td>
<td>deg. 3</td>
<td>deg. 4</td>
</tr>
<tr>
<td>LC-LDPC- I</td>
<td>0</td>
<td>10000</td>
<td>0.3043</td>
<td>0.2770</td>
<td>0.4187</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>8000</td>
<td>0.3043</td>
<td>0.2769</td>
<td>0.4188</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>6000</td>
<td>0.3043</td>
<td>0.2770</td>
<td>0.4187</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>4000</td>
<td>0.3043</td>
<td>0.2769</td>
<td>0.4188</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>2000</td>
<td>0.3043</td>
<td>0.2771</td>
<td>0.4186</td>
</tr>
<tr>
<td>LC-LDPC- II</td>
<td>0</td>
<td>10000</td>
<td>0.3043</td>
<td>0.2770</td>
<td>0.4187</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>8000</td>
<td>0.3352</td>
<td>0.3814</td>
<td>0.2834</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>6000</td>
<td>0.4000</td>
<td>0.6000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>4000</td>
<td>0.4000</td>
<td>0.6000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>2000</td>
<td>0.3999</td>
<td>0.6001</td>
<td>0</td>
</tr>
</tbody>
</table>
Fig. 1. Tanner graphs constructed by the proposed algorithm. Here $\Pi_i$ denotes an interleaver.
Fig. 2. BER performance of LC-LDPC codes obtained from mother codes of rate 0.5 and information length 10000.
Fig. 3. BER performance of LC-LDPC codes obtained from mother codes of rate 0.5 and information length 2000.
Fig. 4. BER performance of LC-LDPC codes obtained from mother codes of rate 0.82 and information length 10000.
Fig. 5. BER performance of LC-LDPC codes obtained from mother codes of rate 0.5 and information length 5400.